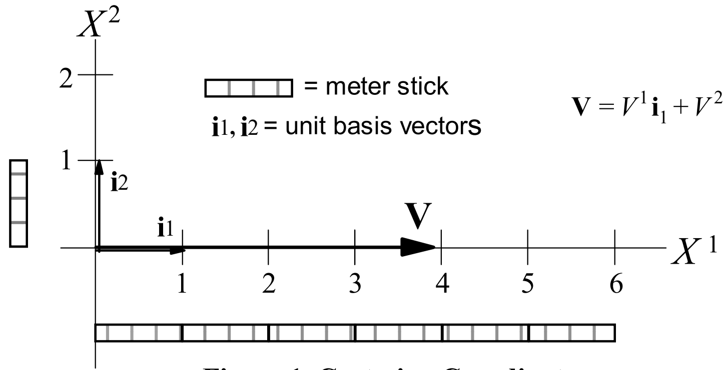
Generalized Coordinates and Metrics

Robert D. Klauber www.quantumfieldtheory.info June 2, 2025

1 Simple Example: Cartesian to Generalized Coordinates

Until further notice, everything here is in a flat space (2D to start with).

1.1 A Vector in Cartesian Coordinates



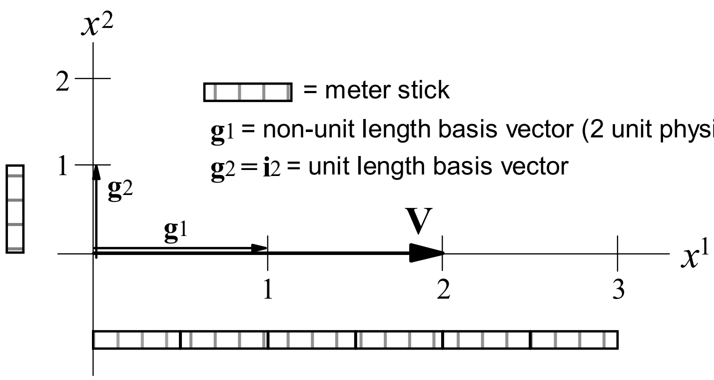
$$\mathbf{V} = V^1\mathbf{i}_1 + V^2\,\mathbf{i}_2 = 4\mathbf{i}_1 + 0\,\mathbf{i}_2 \qquad V^i = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Figure 1. Cartesian Coordinates

$$\begin{aligned}
|\mathbf{V}|^2 &= \mathbf{V}\cdot\mathbf{V} = \left(v^1\mathbf{g}_1\right)\cdot\left(v^1\mathbf{g}_1\right) + \left(V^2\mathbf{i}_2\right)\cdot\left(V^2\mathbf{i}_2\right) = V^1V^1\left(\mathbf{i}_1\cdot\mathbf{i}_1\right) + V^2V^2\left(\mathbf{i}_2\cdot\mathbf{i}_2\right) \\
&= V^1V^1 + V^2V^2 = V^iV^i = \delta_{ij}V^iV^j = \begin{bmatrix} 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = 4^2 = 16 \\
&= \begin{bmatrix} 4 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{i}_1\cdot\mathbf{i}_1 & \\ & \mathbf{i}_2\cdot\mathbf{i}_2 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} V^1 & V^2 \end{bmatrix}\begin{bmatrix} \mathbf{i}_1\cdot\mathbf{i}_1 & \\ & \mathbf{i}_2\cdot\mathbf{i}_2 \end{bmatrix}\begin{bmatrix} V^1 \\ V^2 \end{bmatrix} = g_{ij}V^iV^j
\end{aligned} \tag{1}$$

$$\text{where } g_{ij} = \delta_{ij} = \mathbf{i}_i\cdot\mathbf{i}_j$$

1.2 Same Vector in a Generalized Coordinate System (Curvilinear Coordinate System)



$$\mathbf{V} = v^1\mathbf{g}_1 + v^2\,\mathbf{g}_2 = 2\mathbf{g}_1 + 0\,\mathbf{g}_2 \qquad v^i = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Figure 2. The Same Vector V in a Coordinate System with "Stretched" x^1 Axis.

The vector has the same length regardless of what coordinate system we use (it is invariant). So how do we compute its length (or length squared) in the new system of Fig. 2 that is not Cartesian? In that system **V** is only two units. But those units are not equal to the number of meter sticks, as our labeling on the horizontal axis is not one unit per meter stick.

$$\begin{aligned}
|\mathbf{V}|^2 &= \mathbf{V}\cdot\mathbf{V} = \left(v^1\mathbf{g}_1\right)\cdot\left(v^1\mathbf{g}_1\right) + \left(v^2\mathbf{g}_2\right)\cdot\left(v^2\mathbf{g}_2\right) = v^1v^1\underbrace{\left(\mathbf{g}_1\cdot\mathbf{g}_1\right)}_{\text{label } g_{11}} + v^2v^2\underbrace{\left(\mathbf{g}_2\cdot\mathbf{g}_2\right)}_{\text{label } g_{22}}\left(\neq v^1v^1 + v^2v^2\right) \\
&= g_{11}v^1v^1 + g_{22}v^2v^2 = g_{ij}v^iv^j = (4)v^1v^1 + (1)v^2v^2 = \begin{bmatrix} 2 & 0 \end{bmatrix}\begin{bmatrix} 4 & \\ & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \end{bmatrix}\begin{bmatrix} 8 \\ 0 \end{bmatrix} = 16 \\
&= \begin{bmatrix} 2 & 0 \end{bmatrix}\begin{bmatrix} g_{11} & \\ & g_{22} \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} v^1 & v^2 \end{bmatrix}\begin{bmatrix} g_{11} & \\ & g_{22} \end{bmatrix}\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = g_{ij}v^iv^j \qquad \text{where } g_{ij} = \mathbf{g}_i\cdot\mathbf{g}_j.
\end{aligned} \tag{2}$$

Bottom line #1: The metric components g_{ij} are "fudge factors" that let us use the non-Cartesian (generalized) coordinate system component values v^i to calculate the length (square root of the length squared) of a position vector (or more generally, the magnitude of any vector such as velocity, acceleration, momentum, force, etc.) (See first part of second row of (2).)

Bottom line #2: It turns out each of these "fudge factors" equals the inner product of two of the corresponding basis vectors in the generalized coordinate system.

Problem 1: In Fig. 2, we stretched our horizontal axis by a factor of 2. Draw the equivalent of Fig. 2, if the stretching it were a factor of 4, instead. What are the coordinates in the new coordinate system of the same vector **V**? How long, in terms of meter sticks would the new basis vector $\mathbf{g}_1$ be in order to keep $v^1\mathbf{g}_1$ a vector with length equal to the same number of meter sticks? Repeat all of the steps of equation (2) for this new coordinate system.

Problem 2: Draw the equivalent of Fig. 2, if the stretching it were a factor of ½. What are the coordinates in the new coordinate system of the same vector **V**? How long, in terms of meter sticks would the new basis vector $\mathbf{g}_1$ be in order to keep $v^1\mathbf{g}_1$ a vector with length equal to the same number of meter sticks? Repeat all of the steps of equation (2) for this new coordinate system.

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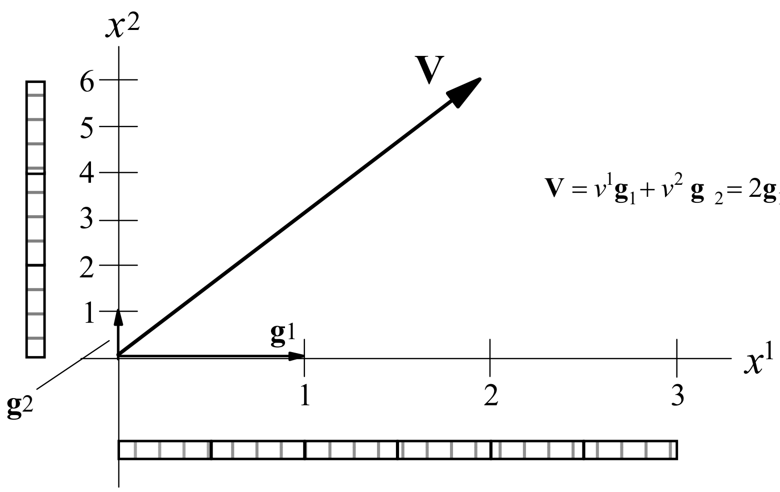
2 More Complicated Examples

One can readily visualize another generalized coordinate system where the vertical axis is also stretched or compressed. We would simply repeat all of the above steps, but for that direction stretching/compressing, in addition to that of the horizontal axis. All of the vertical change in that axis would be reflected in v^2, $\mathbf{g}_2$, and g_{ij}.

Thus, for a more general vector, not simply in the horizontal direction, and a more generalized new coordinate system, we could have something like Fig. 3.



$$\mathbf{V} = v^1\mathbf{g}_1 + v^2\,\mathbf{g}_2 = 2\mathbf{g}_1 + 6\mathbf{g}_2 \qquad v^i = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Figure 3. Vector Not Aligned Along One Axis

Note that the new vector has a component in the x^1 direction of 4 meter sticks, a component in the x^2 direction of 3 meter sticks, so its length is 5 (with length squared of 25). Also, $v^1 = 2$ and $v^2 = 6$. The $\mathbf{g}_1$ basis vector must then have a physical length of 2 meter sticks, so that , v^1 $\mathbf{g}_1$ has a physical length of 4 meter sticks. The $\mathbf{g}_2$ basis vector must then have a physical length of half a meter stick, so that , v^2 $\mathbf{g}_2$ has a physical length of 3 meter sticks.

Then, for the coordinate system of Fig. 3.

$$|\mathbf{V}|^2 = \mathbf{V}\cdot\mathbf{V} = \left(v^1\mathbf{g}_1\right)\cdot\left(v^1\mathbf{g}_1\right) + \left(v^2\mathbf{g}_2\right)\cdot\left(v^2\mathbf{g}_2\right) = v^1v^1\underbrace{\left(\mathbf{g}_1\cdot\mathbf{g}_1\right)}_{\text{label } g_{11}} + v^2v^2\underbrace{\left(\mathbf{g}_2\cdot\mathbf{g}_2\right)}_{\text{label } g_{22}}$$

$$= g_{11}v^1v^1 + g_{22}v^2v^2 = g_{ij}v^iv^j = (4)v^1v^1 + (1)v^2v^2 = \begin{bmatrix} 2 & 6 \end{bmatrix}\begin{bmatrix} 4 & \\ & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 \end{bmatrix}\begin{bmatrix} 8 \\ \frac{3}{2} \end{bmatrix} = 16 + 9 = 4^2 + 3^2 = 25 \quad (3)$$

$$= \begin{bmatrix} 2 & 6 \end{bmatrix}\begin{bmatrix} g_{11} & \\ & g_{22} \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} v^1 & v^2 \end{bmatrix}\begin{bmatrix} g_{11} & \\ & g_{22} \end{bmatrix}\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = g_{ij}v^iv^j \qquad \text{where } g_{ij} = \mathbf{g}_i\cdot\mathbf{g}_j.$$

Again, we see that the metric g_{ij} for a given generalized coordinate system comprises fudge factors for the various components in the vector expressed in the generalized system, such that $g_{ij}v^iv^j$ is the square of the length we would measure physically with meter sticks.

For a generalized system where the shrink or stretch in any axis direction is uniform (the same all along the axis),

$$l^2 = |\mathbf{x}|^2 = g_{ij}x^ix^j \qquad \mathbf{x}= \text{position vector} \quad x^i = \text{components in generalized coordinates} . \tag{4}$$

3 Non-Unform Generalized Coordinates

The above examples had the same stretch or compression of the coordinate grid lines all along a given axis. More generally, one can have different amounts of stretch/compression at different points along an axis, as in the lefthand diagram of Fig. 4.

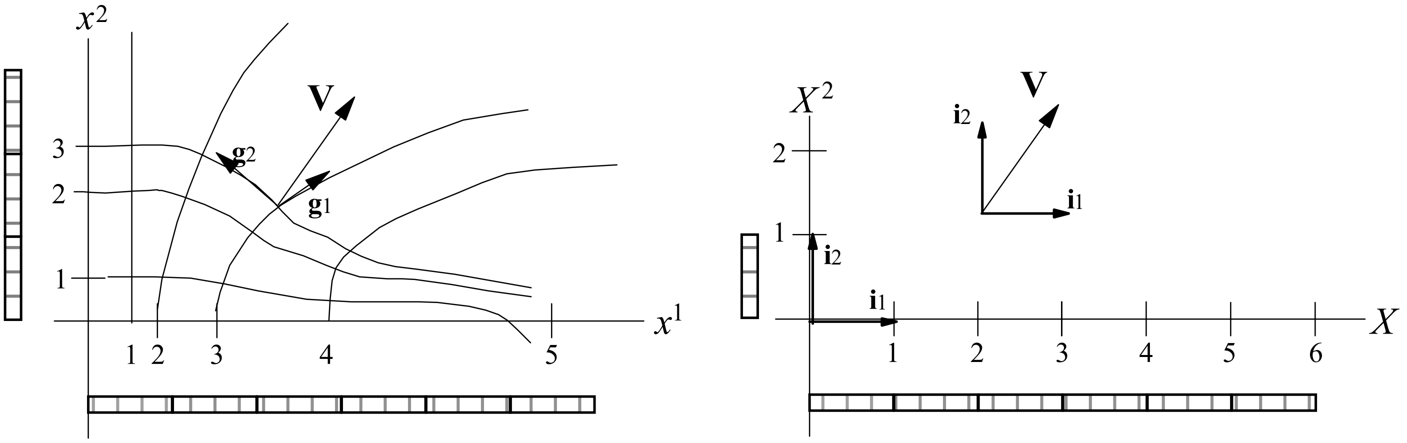


Figure 4. Non-uniformly Spaced Grid Lines vs Cartesian Grids

For our vector **V**, in order to determine its magnitude (squared) in the lefthand diagram, we need to know its components in terms of the local coordinate system at the point where **V** originates. If we know the directions and magnitudes of $\mathbf{g}_1$ and $\mathbf{g}_2$ at that point, and the direction and magnitude of **V** in physical space, we can calculate what the coordinates of **V** are in the generalized coordinate grid, i.e., v^1 and v^2. Then, as before, only now just in one spot in space,

$$\mathbf{V} = v^1\mathbf{g}_1 + v^2\mathbf{g}_2 \quad \text{at the point where } \mathbf{V} \text{ is.}$$

$$|\mathbf{V}|^2 = \mathbf{V}\cdot\mathbf{V} = \left(v^1\mathbf{g}_1\right)\cdot\left(v^1\mathbf{g}_1\right) + \left(v^2\mathbf{g}_2\right)\cdot\left(v^2\mathbf{g}_2\right) = v^1v^1 \underbrace{\left(\mathbf{g}_1\cdot\mathbf{g}_1\right)}_{\text{label } g_{11}} + v^2v^2 \underbrace{\left(\mathbf{g}_2\cdot\mathbf{g}_2\right)}_{\text{label } g_{22}} \tag{5}$$

$$= g_{11}v^1v^1 + g_{22}v^2v^2 = g_{ij}v^iv^j \qquad \text{where } \boxed{g_{ij} = \mathbf{g}_i\cdot\mathbf{g}_j} .$$

We often ignore the basis vectors $\mathbf{g}_1$ and $\mathbf{g}_2$, and just focus on the matric g_{ij} and the components v^i, i.e., use just the last part of (5).

Now in Fig. 4, our basis vectors, and thus our metric (each component of which is an inner product between two basis vectors) are different at different points in space. That is, they are functions of the generalized coordinates x^i. So, the most general way to write the magnitude squared of a vector in 2D space is

$$\boxed{|\mathbf{V}|^2 = g_{ij}\left(x^1,x^2\right)v^iv^j} \qquad \text{where typically, also } v^i = v^i\left(x^1,x^2\right) . \tag{6}$$

Note the differential position vector $d\mathbf{x}$, is a vector, so for that particular vector, (6) becomes

$$\boxed{|d\mathbf{x}|^2 = g_{ij}dx^idx^j} \qquad \text{where } g_{ij} = g_{ij}\left(x^1,x^2\right) , \tag{7}$$

which gives us the distance measured in meter sticks, for an infinitesimal displacement in a generalized coordinate system.

(7) is often expressed in terms of what is called the line element,

$$|d\mathbf{x}|^2 = g_{11}dx^1dx^1 + g_{22}dx^2dx^2 . \tag{8}$$

This is the length (in meters) squared of a line of differential length, so the name is appropriate.

4 Higher Dimensional Spaces

4.1 Purely Spatial Spaces

We can generalize (6) to (8) to spaces of any number of spatial dimensions, 3D, 10D, etc.

$$\begin{aligned} |\mathbf{v}|^2 &= g_{ij}\left(x^i\right)v^i v^j \quad i,j=1,\ldots,N \quad \text{where } N = \text{number of dimensions} \\ |d\mathbf{x}|^2 &= g_{ij}dx^i dx^j = g_{11}dx^1dx^1 + g_{22}dx^2dx^2 + \ldots + g_{NN}dx^N dx^N . \end{aligned} \tag{9}$$

Problem 3. A polar coordinate system is a generalized (non-Cartesian) coordinate system with $x^i = [x^1, x^2]^T = [r, \phi]^T$. Sketch the gird lines for this system. Express the line element for it. From that, deduce the metric. Does it depend on location in space?

4.2 Flat Spacetime

In 4D spacetime, we deal with the invariant interval s, which is a distance, often just infinitesimal ds, in spacetime between two spacetime points (two events). In special relativity, in Minkowski coordinates (Cartesian plus time), this was just

$$\begin{gathered} ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{00}dx^0dx^0 + g_{11}dx^1dx^1 + g_{22}dx^2dx^2 + g_{33}dx^3dx^3 \;\;(\text{line element}) \\ \text{metric } g_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad dx^0 = cdt. \end{gathered} \tag{10}$$

Relation (10) holds for inertial systems. It is good for non-accelerating and non-gravitational (flat) spaces.

5 Curved Spaces

All of the above was derived for flat spaces (or flat spacetimes) and orthogonal grid line systems.. Indeed, (9) is really just the Pythagorean theorem, where, for example, $g_{11}dx^1\, dx^1$ is just the length squared in the x^1 direction, with similar conclusion for the other directions.

But, we can still use all of what we have deduced for flat spaces locally, in curved spaces. That is, as long as we are talking an infinitesimal distance, and infinitesimal distance appears flat locally, just as the Earth seems flat to us locally (effectively infinitesimal distance compared to the Earth circumference.)

Thus, (9) holds in a curved space, such as the surface of a sphere in 2D.

$$|d\mathbf{x}|^2 = g_{ij}dx^i dx^j = g_{11}dx^1dx^1 + g_{22}dx^2dx^2 + \ldots + g_{NN}dx^N dx^N \quad \left(g_{ij} = g_{ij}\left(x^i\right)\right). \tag{11}$$

Similarly, for curved spacetime, (10) holds, except the metric is not constant, but depends on x^μ.

$$\begin{gathered} ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{00}dx^0dx^0 + g_{11}dx^1dx^1 + g_{22}dx^2dx^2 + g_{33}dx^3dx^3 \;\;(\text{line element}) \\ \text{metric } g_{\mu\nu} = g_{\mu\nu}\left(x^\mu\right) = \begin{bmatrix} g_{00}\left(x^\mu\right) & & & \\ & g_{11}\left(x^\mu\right) & & \\ & & g_{22}\left(x^\mu\right) & \\ & & & g_{33}\left(x^\mu\right) \end{bmatrix} \end{gathered} \tag{12}$$

6 Covariant (One-Form) Components

6.1 Purely Spatial Space

If we define covariant components as

$$v_i = g_{ij}v^j , \tag{13}$$

then, in (11), even when the metric is a function of position, and even for curved spaces,

$$
\begin{aligned}
|\mathbf{V}|^2 &= g_{ij} v^i v^j = g_{ij} v^j v^i = v_i v^i \\
dl^2 = |d\mathbf{x}|^2 &= g_{ij} dx^i dx^j = g_{ij} dx^j dx^i = dx_i dx^i .
\end{aligned}
\tag{14}
$$

Both quantities on the left in (14) are invariant, the same in any generalized coordinate system.

6.2 Spacetime

In 4D spacetime, then, even in curved spaces where $g_{\mu\nu}$ is a function of spacetime location x^μ, locally, we define covariant components as

$$u_\mu = g_{\mu\nu} u^\nu , \tag{15}$$

so,

$$(u)^2 = g_{\mu\nu} u^\nu u^\nu = u_\mu u^\mu \tag{16}$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx_\mu dx^\mu . \tag{17}$$

7 Solving Real-World Problems

In principle, any grid lines we wish to set up in purely spatial spaces, or in spacetime, will work to solve a problem. But, as we always find in physics, particular choices for coordinate systems can make solving either a whole lot easier, or even make otherwise intractable problems, tractable.

For example, a spherical coordinate system is preferred for problems with spherical symmetry, like the gravitational field around a planet, star, or black hole. A cylindrical system is preferred for finding the stresses in a cylindrical rod, like the driveshaft in a car. In superstring theory, a toroidal coordinate system can be helpful when strings wind themselves into toroids.

In most cases, the invariant quantities, like vector magnitude, distance (spatial systems), or interval (relativity) are important. They are the same regardless of the coordinate system we choose, so we are at liberty to pick the one that helps most in solving the problem.

8 Non-orthogonal Coordinate Grids

8.1 Orthogonal Coordinate Grids (Systems)

In all that we have done to here, for flat or curved spaces, we have focused on coordinate grid lines that were orthogonal. Yes, they were not generally equally spaced (in physical space), and not generally straight lines, as in Fig. 4, but each intersection of grid lines was at a right angle.

This meant our generalized basis vectors $\mathbf{g}_i$ were orthogonal, so in (5) we had (with similar results in spacetime)

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j \quad = 0 \quad \text{for } i \neq j . \tag{18}$$

Hence, our metrics to date have all been of diagonal form (whether constant or dependent on position), as in (10) and (12).

8.2 Non-Orthogonal Purely Spatial Coordinate Grids (Systems)

Fig. 5 shows two possible 2D spatial coordinate systems where the generalized basis vectors $\mathbf{g}_1$ and $\mathbf{g}_2$ are not orthogonal. The system on the left has uniform spacing between lines; the system on the right, non-uniform.

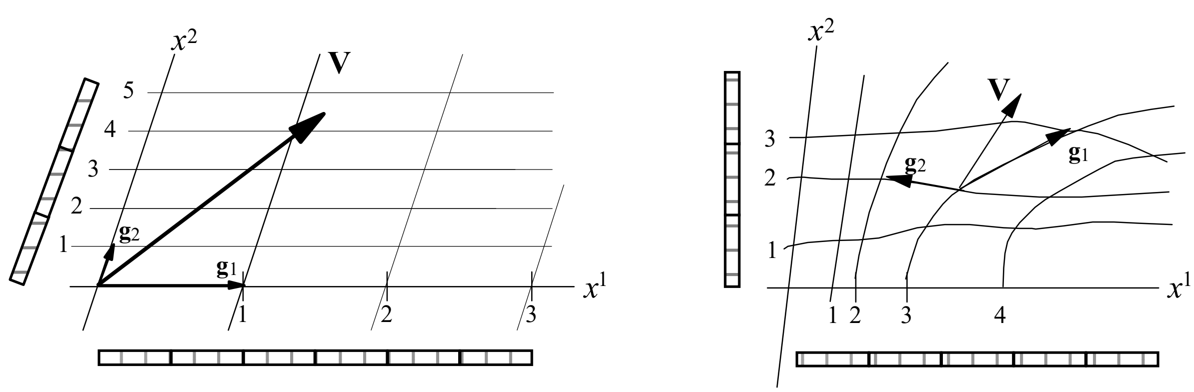


Figure 5. Non-orthogonal Grid Lines Coordinate Systems

For the left-hand diagram of Fig. 5, we would have a metric, where all components are constant throughout the space of

$$g_{ij}=\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \quad g_{11}, g_{12}, g_{21}, g_{22} \text{ all constant numbers, } \; g_{12}=\mathbf{g}_1\cdot\mathbf{g}_2=\mathbf{g}_2\cdot\mathbf{g}_1=g_{21}\neq 0\,. \tag{19}$$

Note that $g_{12} = g_{21}$, and this is generally true for metrics, i.e., $g_{ij} = g_{ji}$. Metrics are symmetric. The transpose of a metric equals the metric.

For the right-hand diagram of Fig. 5, we would have a metric, where all components vary throughout the space of

$$g_{ij}(x^k)=\begin{bmatrix} g_{11}(x^k) & g_{12}(x^k) \\ g_{21}(x^k) & g_{22}(x^k) \end{bmatrix} \quad g_{11}, g_{12}, g_{21}, g_{22} \text{ functions of position, } \; g_{12}=\mathbf{g}_1\cdot\mathbf{g}_2=\mathbf{g}_2\cdot\mathbf{g}_1=g_{21}\neq 0 \tag{20}$$

8.3 Non-orthogonal Metrics in Spacetime

Most problems in relativity, special (with constant Minkowski metric) and general (with non-constant generalized metric), employ diagonal metrics. For such problems, we could use any spacetime gridline we choose, but it is generally easier with diagonal metrics, so we choose systems with grid lines that are orthogonal, even though they are not uniformly spaced (like Fig. 4, but for spacetime).

However, for some cases, like the solution to Einstein's equation for a rotating planet, star, or black hole, the only way they can be solved is with non-orthogonal systems, i.e., off-diagonal terms in the metric. The most famous of these solutions is the Kerr metric, which, in fact, is the solution for the gravitationally induced field around rotating spherical body. It was a challenging problem for many years until Roy Kerr deduced the metric for the field.

9 Finding $\mathbf{g}_i$ Mathematically

9.1 Visualizing Basis Vectors and Components for Given Grid Line Coordinates

Note in Fig. 2, that stretching our coordinates on the horizontal axis by a factor of two, increased the length of our basis vector from 1 meter ($\mathbf{i}_1$ in Fig. 1) to 2 meters ($\mathbf{g}_1$ in Fig. 2). The basis vectors $\mathbf{g}_i$ increase in length by the same factor as the coordinates are stretched. This can also be seen in Probs. 1 and 2, as well as the vertical axis in Fig. 3.

In tandem with this, the contravariant vector components were reduced by one over the stretching factor. In Fig. 2, v^1 is half of what V^1 was in Fig. 1, where the coordinates were stretched by a factor of 2. This behavior repeats in Probs. 1 and 2.

Bottom line: The length of a $\mathbf{g}_i$ basis vector in meter sticks is equal to the factor by which we stretch the physical length measuring coordinate grid in the $\mathbf{g}_i$ direction to the generalized coordinate grid. The corresponding generalized component v^i of a vector is the physical length coordinate divided by that factor.

The generalized coordinate varies in an opposite, a contrary, way from the physical length. If the grid gets larger, it gets smaller. Thus, we call v^i the contravariant component.

Similar logic can show that the covariant component v_i varies in a covariant fashion with grid stretch. If the grid gets larger, the covariant component also gets larger. "Contra" is against stretch/compression amount. "Covariant" is with stretch/compression amount. Indeed, if we have the invariant vector length squared

$$|\mathbf{V}|^2 = v_i v^i\,, \tag{21}$$

as we stretch gridlines, v^i gets smaller, so v_i must get proportionately larger. It must vary covariantly with grid line spacing.

Note from Figs. 2 and 3 (as well as Probs. 1 and 2) that a $\mathbf{g}_i$ basis vector gets longer when its associated grid line spacing gets larger; and shorter, when the spacing gets smaller. Hence, it varies covariantly with the grid line spacing. So, we say $\mathbf{g}_i$ is a covariant basis vector. Note that we have a contravariant vector component for each covariant basis vector, $\mathbf{v} = v^1\mathbf{g}_1 + v^2\mathbf{g}_2$.

Similarly, we can expect to have contravariant basis vectors $\mathbf{g}^i$, such that

$$\mathbf{v} = v_1\mathbf{g}^1 + v_2\mathbf{g}^2\,, \tag{22}$$

where v_i varies covariantly with grid line spacing, and $\mathbf{g}^i$ varies contravariantl.

9.2 Finding Basis Vectors Mathematically

We will now state a math relation for determining $\mathbf{g}_i$, without proof, and then see how it works for the examples we've studied. For this, consider **P** the position vector of a point in our space, which can be expressed, for flat space, in terms of Cartesian coordinates.

$$\mathbf{P} = X^1\mathbf{i}_1 + X^2\mathbf{i}_2 + \ldots X^N\mathbf{i}_N \quad N = \text{number of dimensions in the space; } X^i = \text{ Cartesian coordinates}. \tag{23}$$

Then, we state (and will illustrate shortly)

$$\mathbf{g}_1 = \frac{\partial \mathbf{P}}{\partial x^1} \quad \mathbf{g}_2 = \frac{\partial \mathbf{P}}{\partial x^2} \quad \ldots \quad \text{or} \qquad \mathbf{g}_i = \frac{\partial \mathbf{P}}{\partial x^i}. \tag{24}$$

Let's look at Figs. 1 and 2, and apply (24). Note from those figures,

$$X^1 = f\left(x^1\right) = 2x^1 \qquad X^2 = g\left(x^2\right) = x^2, \tag{25}$$

so, (24) becomes

$$\begin{aligned} \mathbf{g}_1 &= \frac{\partial \mathbf{P}}{\partial x^1} = \frac{\partial}{\partial x^1}\left(X^1\mathbf{i}_1 + X^2\mathbf{i}_2\right) = \frac{\partial}{\partial x^1}\left(2x^1\mathbf{i}_1 + x^2\mathbf{i}_2\right) = 2\mathbf{i}_1 \\ \mathbf{g}_2 &= \frac{\partial \mathbf{P}}{\partial x^2} = \frac{\partial}{\partial x^2}\left(X^1\mathbf{i}_1 + X^2\mathbf{i}_2\right) = \frac{\partial}{\partial x^2}\left(2x^1\mathbf{i}_1 + x^2\mathbf{i}_2\right) = \mathbf{i}_2. \end{aligned} \tag{26}$$

We can generalize (26) for any generalized basis vector in any number of dimensions, for purely spatial spaces or for spacetime.

$$\boxed{\text{Pure spatial} \quad \mathbf{g}_i = \frac{\partial \mathbf{P}}{\partial x^i} \quad i = 1,\ldots,N \qquad\qquad \text{Spacetime} \quad \mathbf{g}_\mu = \frac{\partial \mathbf{P}}{\partial x^\mu}} \tag{27}$$

Problem 7. Using what you learned from Prob. 3, find g^{ij} for a polar coordinate system. Then, for a vector $v^i = [3, 2]^T$ find the covariant components v_i in terms of r. For that vector, show that its length squared is given by both $g_{ij}v^iv^j$ and $g^{ij}v_iv_j$.

Problem 8. Using what you learned from Sect. 9, find $\mathbf{g}_1 = \mathbf{g}_r$ and $\mathbf{g}_2 = \mathbf{g}_\phi$ for polar coordinates. Note that in the Cartesian system $\mathbf{P} = X^1\mathbf{i}_1 + X^2\mathbf{i}_2 = r\cos\phi\,\mathbf{i}_1 + r\sin\phi\,\mathbf{i}_2$. In the figure you drew for Prob. 3, at the point $r = 3$ and $\phi = \pi/2$, draw in $\mathbf{g}_1 = \mathbf{g}_r$ and $\mathbf{g}_2 = \mathbf{g}_\phi$. At the point $r = 2$ and $\phi = \pi/4$, draw in $\mathbf{g}_1 = \mathbf{g}_r$ and $\mathbf{g}_2 = \mathbf{g}_\phi$. Does $g_{ij} = \mathbf{g}_i \bullet \mathbf{g}_j$ in both cases? If so, show it.

Problem 9. Consider a polar coordinate system centered on a vortex (like a hurricane or water swirling down a sink drain) and assume we live in a Newtonian (not relativistic) world. The physically measured velocity of the fluid is dependent on position. Fluid molecules far from the center travel at slower speeds than molecules close to the center. The generalized coordinates of velocity depends on location, i.e., on r and ϕ, expressed as $v^i = v^i(r, \phi) = v^i(x^1, x^2)$. At the point $r = 3$ and $\phi = \pi/2$ velocity is, in polar coordinates, $v^i(3, \pi/2) = [0, 2/3]^T = 0\,\mathbf{g}_r + 2/3\,\mathbf{g}_\phi$. What direction in space does this vector point in? Using g_{ij} find the magnitude of this vector (its speed). Draw this vector in the figure you drew for Prob. 3.

Problem 10. For the same vortex of Prob. 9, and the second point ($r = 2$ and $\phi = \pi/4$) of Prob. 8, the velocity, in generalized coordinates is $v^i(2, \pi/4) = [0, 1/3]^T = 0\,\mathbf{g}_r + 1/3\,\mathbf{g}_\phi$ What direction in space does this vector point in? Using g_{ij} find the magnitude of this vector (its speed). Draw this vector in the figure you drew for Prob. 3.

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10 The Covariant Derivative

10.1 The Same Vector Field in Cartesian and Generalized Coordinate Systems

Consider a vector field in a 2D plane expressed in Cartesian coordinates plus some generalized coordinates system. The vector field has different vector values at different points in the 2D space.

$$\mathbf{V}=V^1\mathbf{i}_1+V^2\mathbf{i}_2=V^i\mathbf{i}_i \qquad V^1=V^1\left(X^1,X^2\right) \qquad V^2=V^2\left(X^1,X^2\right) \qquad \mathbf{i}_1,\mathbf{i}_2 \text{ constant over all space}$$
$$\text{Often expressed as } \begin{bmatrix} V^1 \\ V^2 \end{bmatrix} \text{ or } V^i \text{ or } V^i\left(X^1,X^2\right) \tag{28}$$

$$\mathbf{V}=v^1\mathbf{g}_1+v^2\mathbf{g}_2=v^i\mathbf{g}_i \qquad v^1=v^1\left(x^1,x^2\right) \qquad v^2=v^2\left(x^1,x^2\right) \qquad \mathbf{g}_1=\mathbf{g}_1\left(x^1,x^2\right) \qquad \mathbf{g}_2=\mathbf{g}_2\left(x^1,x^2\right)$$
$$\text{Often expressed as } \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \text{ or } \mathrm{v}^i \text{ or } \mathrm{v}^i\left(x^1,x^2\right) \tag{29}$$

The same vector field **V** is expressed in two different coordinate systems in (28) and (29). It varies with location (like a velocity field of molecules on the surface of water in a river, for example).

10.2 Derivatives with Respect to Coordinates for Cartesian Coordinates

Let's take the derivative with respect to a given coordinate of (28). e.g., X^1,

$$\frac{\partial\mathbf{V}}{\partial X^1}=\frac{\partial}{\partial X^1}\left(V^1\mathbf{i}_1+V^2\mathbf{i}_2\right)=\frac{\partial V^1}{\partial X^1}\mathbf{i}_1+V^1\underbrace{\frac{\partial\mathbf{i}_1}{\partial X^1}}_{=0}+\frac{\partial V^2}{\partial X^1}\mathbf{i}_2+V^2\underbrace{\frac{\partial\mathbf{i}_2}{\partial X^1}}_{=0}=\frac{\partial V^1}{\partial X^1}\mathbf{i}_1+\frac{\partial V^2}{\partial X^1}\mathbf{i}_2=\frac{\partial V^i}{\partial X^1}\mathbf{i}_i$$
$$\text{Or often expressed as just } \frac{\partial V^i}{\partial X^1} \text{ or } V^i{}_{,1} \tag{30}$$

Or more generally, for any coordinate derivative

$$\frac{\partial\mathbf{V}}{\partial X^j}=\frac{\partial V^i}{\partial X^j}\mathbf{i}_i+V^i\frac{\partial\mathbf{i}_i}{\partial X^j}=\frac{\partial V^i}{\partial X^j}\mathbf{i}_i \qquad \text{Or often expressed as just } \frac{\partial V^i}{\partial X^j} \text{ or } V^i{}_{,j}\,. \tag{31}$$

10.3 Derivatives with Respect to Coordinates for Generalized Coordinates

Let's repeat the steps of (30) and (31) i.e., take derivatives with respect to coordinates, for the generalized coordinate system of (29). Note that now the basis vectors $\mathbf{g}_i$ are not constant. Unlike the Cartesian basis vectors $\mathbf{i}_i$, they vary with position.

$$\frac{\partial\mathbf{V}}{\partial x^1}=\frac{\partial}{\partial x^1}\left(v^1\mathbf{g}_1+v^2\mathbf{g}_2\right)=\frac{\partial v^1}{\partial x^1}\mathbf{g}_1+v^1\underbrace{\frac{\partial\mathbf{g}_1}{\partial x^1}}_{\substack{\text{NOT}\\=0}}+\frac{\partial v^2}{\partial x^1}\mathbf{g}_2+v^2\underbrace{\frac{\partial\mathbf{g}_2}{\partial x^1}}_{\substack{\text{NOT}\\=0}}=\frac{\partial\left(v^i\mathbf{g}_i\right)}{\partial x^1}\,. \tag{32}$$

This can be re-expressed as

$$\begin{aligned}\frac{\partial\mathbf{V}}{\partial x^1}&=\frac{\partial v^1}{\partial x^1}\mathbf{g}_1+\frac{\partial v^2}{\partial x^1}\mathbf{g}_2+v^1\underbrace{\frac{\partial\mathbf{g}_1}{\partial x^1}}_{\substack{\text{NOT}\\=0}}+v^2\underbrace{\frac{\partial\mathbf{g}_2}{\partial x^1}}_{\substack{\text{NOT}\\=0}}=\frac{\partial v^i}{\partial x^1}\mathbf{g}_i+v^i\underbrace{\frac{\partial\mathbf{g}_i}{\partial x^1}}_{\Gamma^k_{i1}\mathbf{g}_k}\\&=\frac{\partial v^i}{\partial x^1}\mathbf{g}_i+v^i\Gamma^k_{i1}\mathbf{g}_k=v^i_{,1}\mathbf{g}_i+v^i\Gamma^k_{i1}\mathbf{g}_k\,.\end{aligned} \tag{33}$$

$\Gamma^k{}_{i1}$ is called a <u>Christoffel symbol</u> (after the man who developed them) or a <u>connection</u>. It results because the $\mathbf{g}_i$ basis vectors are not constants, so we can't take their derivatives as zero, like we did in (30). We will consider its mathematical form shortly.

For now, we can generalize (33) to derivatives with respect to any coordinate (there are only two coordinates here, but in general, there can be more).

$$\frac{\partial\mathbf{V}}{\partial x^j}=\frac{\partial v^i}{\partial x^j}\mathbf{g}_i+v^i\Gamma^k_{ij}\mathbf{g}_k=v^i_{,j}\,\mathbf{g}_i+v^i\Gamma^k_{ij}\mathbf{g}_k \tag{34}$$

$$\text{Or often expressed as just } \frac{Dv^i}{Dx^j} \text{ or } v^i{}_{;j}\,, \tag{35}$$

where the D symbol for the derivative is called the covariant derivative.[1]

Problem 16. Make a sketch showing, in polar coordinates, how the $\mathbf{g}_r$ basis vector (think $\mathbf{g}_1$ in (33) changes direction as we vary ϕ. In other words, the derivative of that vector with respect to ϕ is not simply a change along the original direction of that vector. So, we can't expect both terms in (34) to be simply numbers multiplied by the original vector $\mathbf{g}_r$. The derivative must incorporate other directions, i.e., other $\mathbf{g}_i$. And (34) does just that. The Christoffel symbols tell us how much of each other basis vector plays as part of the change in $\mathbf{g}_r$.

10.4 Generalizing

The relations (34) and (35) are valid for any number of dimensions and also includes 4D spacetime (one dimension is time, not space). So, we can write

$$\frac{\partial \mathbf{V}}{\partial x^\nu} = \frac{\partial v^\mu}{\partial x^\nu}\mathbf{g}_\mu + v^\mu \Gamma^\rho_{\mu\nu}\mathbf{g}_\rho = v^\mu_{,\nu}\,\mathbf{g}_\mu + v^\mu \Gamma^\rho_{\mu\nu}\mathbf{g}_\rho \tag{36}$$

We can switch the dummy coordinates μ and ρ in the last term of (36) to get a more useful form of the covariant derivative

$$\frac{\partial \mathbf{V}}{\partial x^\nu} = \frac{\partial v^\mu}{\partial x^\nu}\mathbf{g}_\mu + v^\rho \Gamma^\mu_{\rho\nu}\mathbf{g}_\mu = \left(\frac{\partial v^\mu}{\partial x^\nu} + v^\rho \Gamma^\mu_{\rho\nu}\right)\mathbf{g}_\mu = \left(v^\mu_{,\nu} + v^\rho \Gamma^\mu_{\rho\nu}\right)\mathbf{g}_\mu$$

$$\text{Often expressed as just } \quad \frac{Dv^\mu}{Dx^\nu} = \frac{\partial v^\mu}{\partial x^\nu} + v^\rho \Gamma^\mu_{\rho\nu} \quad \text{or} \quad v^\mu_{;\nu} = v^\mu_{,\nu} + v^\rho \Gamma^\mu_{\rho\nu}. \tag{37}$$

10.5 Derivatives with Respect to Proper Time

In GR, whenever we take derivatives, because the $\mathbf{g}_\mu$ are usually not constant, we need to use the covariant derivative, not the partial derivative.

Note that if we take a derivative with respect to proper time, we need to find it as follows.

$$\frac{dv^\mu}{d\tau} \xrightarrow[\text{relativity}]{\text{general}} \frac{Dv^\mu}{D\tau} = \frac{Dv^\mu}{Dx^\nu}\frac{dx^\nu}{d\tau} = v^\mu_{;\nu}\frac{dx^\nu}{d\tau} = \left(\frac{\partial v^\mu}{\partial x^\nu} + v^\rho \Gamma^\mu_{\rho\nu}\right)\frac{dx^\nu}{d\tau} \tag{38}$$

10.6 Mathematical Form for Christoffel Symbols

The Christoffel symbols can be found from the metric, via the relation below, which we are about to derive.

$$\Gamma^\sigma_{\mu\nu} = \tfrac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) \tag{39}$$

Note that, formally, there are two kinds of Christoffel symbols called the first and second kind of Christoffel symbol, and two different notations employed in the literature.

$$\begin{array}{lll} \text{First kind} & \Gamma_{\sigma\mu\nu} = \tfrac{1}{2}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) & \text{alternate notation } \{\mu\nu,\sigma\} \\ \text{Second kind} & \Gamma^\sigma_{\mu\nu} = \tfrac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) & \text{alternate notation } \left\{ \begin{matrix} \sigma \\ \mu\nu \end{matrix} \right\} \end{array} \tag{40}$$

We will use, as most authors do, the Γ notation rather than the {}, and for us, the term "Christoffel symbol" will, unless specified otherwise, refer to the second kind.

The derivation of (39) can be found in the appendix.

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[1] This is different from the gauge covariant derivative (often called just the "covariant derivative") of quantum field theory. Note though, there are similarities. Both entail a partial derivative plus another term. The QED covariant derivative has form $\frac{D\psi}{Dx^\mu} = \frac{\partial \psi}{\partial x^\mu} + i(-e)A_\mu \psi = \left(\frac{\partial}{\partial x^\mu} + i(-e)A_\mu\right)\psi$.

11 The Geodesic Equation

11.1 Derivation of Geodesic Equation

In GR, when we take derivatives, we always have to do so with the covariant derivative (36), rather than the partial derivative we used in the Minkowski coordinates of special relativity, or the Cartesian coordinates of Newtonian theory.

Note that in Newtonian theory, when there is no force on an object, the 2nd law says the velocity is constant, so the time derivative of velocity is constant. We can presume that a comparable law exists in relativity theory, which reduces to the Newtonian version at low velocity. Recall that in GR, gravity is not a force. An object in free fall (on a geodesic path in spacetime) feels no force. So, we can presume a comparable GR expression to the second law, for zero force, using the covariant time derivative rather than the regular time derivative, and proper time rather than its low speed approximation, clock time.

$$\begin{aligned}&\text{Newton, for zero force:}\quad ma^i = m\frac{dv^i}{dt} = 0 \quad\rightarrow\quad \frac{dv^i}{dt} = 0\\ &\text{Einstein, " \quad " \quad " :}\quad \frac{Du^\mu}{D\tau} = 0 \quad \text{At low speed, for 3D velocity equals above.}\end{aligned} \tag{41}$$

$$\begin{aligned}\frac{Du^\mu}{D\tau} &= \frac{Du^\mu}{Dx^\nu}\frac{dx^\nu}{d\tau} = u^\mu_{\;;\nu}\frac{dx^\nu}{d\tau} = \left(\frac{\partial u^\mu}{\partial x^\nu} + u^\rho \Gamma^\mu_{\rho\nu}\right)\frac{dx^\nu}{d\tau} = \frac{\partial u^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau} + u^\rho \Gamma^\mu_{\rho\nu}\frac{dx^\nu}{d\tau}\\ &= \frac{du^\mu}{d\tau} + \Gamma^\mu_{\rho\nu}u^\rho\frac{dx^\nu}{d\tau} = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\rho\nu}u^\rho u^\nu = 0\end{aligned} \tag{42}$$

The geodesic equation in spacetime, re-written in terms of 4D position, instead of 4-velocity, is

$$\frac{Du^\mu}{D\tau} = \boxed{\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0} \qquad \tau = \text{proper time on object}\ . \tag{43}$$

11.2 Usefulness of the Geodesic Equation

With the solution $g_{\mu\nu}$ (and hence the Christoffel symbol via (39)), one can use (43) to describe the motion, in terms of the generalized coordinates $x^\mu(\tau)$, of a particle in free fall (inertial, feeling no force) in the gravitational field of the problem at hand.

As we will show in a class shortly, the spacetime interval (which equals the negative of the proper time on a particle) for a particle traveling along a geodesic path is minimal. Any other path has a greater interval.

11.3 Good for Purely Spatial Spaces As Well

Although we haven't derived it above, the same relation (43) works in purely spatial spaces as well as spacetime. Simply change the Greek indices to Roman and use λ in place of proper time τ, where λ is the distance traveled along a geodesic path (the shortest path between two points in a purely spatial space).

Problem 4. Express (43) in terms of the 4-velocity u^μ.

Problem 5. Determine the Christoffel symbols in a Minkowski coordinate system (special case of generalized coordinates, with Cartesian coordinates in 3D and physical time (times c) as the 4th coordinate). Explain how you got your answer.

11.4 Minkowski and Newtonian Limits of the Geodesic Equation

Note, in (43), that for a Minkowski coordinate system, the Christoffel symbols are all zero, so we have simply (for systems with no e/m force acting on the particle)

$$\frac{d^2x^\mu}{d\tau^2} = 0\ . \tag{44}$$

For low speeds, where proper time τ on the particle is effectively time t on clocks in the coordinate system, the 3D part of (44) reduces to Newton's first law.

$$\frac{d^2x^i}{dt^2}=0 \quad \to \quad m\frac{d^2x^i}{dt^2}=0 \quad \to \quad ma^i=0 \quad \to mv^i = \text{constant}. \tag{45}$$

11.5 Motion with Force Acting

Gravity in GR is not a force, but a curvature of space (reflected in the non-zero Riemann tensor (50) shown below). In the geodesic equation, gravity arises from the non-zero Christoffel symbols of curved space. To see this, rearrange (43) as

$$\frac{d^2x^\mu}{d\tau^2}=-\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}. \tag{46}$$

The term on the right affects the path $x^\mu(\tau)$ of the particle in coordinate space. For a curved space, this is effectively what, in classical non-relativistic theory, we call a gravitational force. Note particularly, that there is no mass term in (46). This is because, for gravity, all masses "fall" (travel a geodesic path) in the same way.

But electricity and magnetism can still apply force, other than gravity, to a particle. This is incorporated by adding an e/m force term to (43) to get (47). The term on the RHS there is simply the e/m force (static electric field force plus Lorentz force from a magnetic field on a moving charge) in 4D relativistic notation.

$$ma^\mu = m\frac{Du^\mu}{D\tau}=m\left(\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\right)=qF^{\mu\beta}\frac{dx_\beta}{d\tau} \qquad F^{\mu\beta} = \text{electromagnetic field tensor}. \tag{47}$$

Problem 6. Express (47) in terms of the 4-velocity u^μ. Does this look like what you (hopefully) learned for the equation of motion for a force on a charged particle due to an e/m field? Use the form of $F^{\mu\nu}$ expressed in terms of the electric and magnetic field components and express the result for low velocities ($v << c$), in flat spacetime (Minkowski metric), for only the 3D part of the velocity. Look familiar?

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12 Einstein's Equation and Finding the Metric for a Given GR Problem

12.1 Einstein's Equation Stated

The Einstein field equation, like the Schrödinger, Newton second law, and Maxwell equations, cannot be derived. It must be deduced, essentially guessed at, and then tested against real world experiment to determine if it represents Nature correctly. This is what Einstein did, and for over 100 years now, it has stood firm through a great many empirical tests.

The Einstein field equation, with definitions below it for the various geometric quantities on the left-hand side, is

$$G_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}=8\pi GT_{\mu\nu} \qquad G_{\mu\nu}= \text{Einstein tensor} \tag{48}$$

$$\text{Ricci tensor} = R_{\mu\nu}=R^\lambda_{\mu\lambda\nu}=g^{\lambda\rho}R_{\rho\mu\lambda\nu} \quad \text{Ricci curvature (or Ricci scalar)} = R=R^\mu_\mu=g^{\mu\nu}R_{\mu\nu} \tag{49}$$

$$\text{Riemann tensor} = R^\rho{}_{\sigma\mu\nu}=\partial_\mu\Gamma^\rho_{\nu\sigma}-\partial_\nu\Gamma^\rho_{\mu\sigma}+\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}-\Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{50}$$

$$\text{Christoffel symbol} = \Gamma^\sigma_{\mu\nu}=\tfrac{1}{2}g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho}+\partial_\nu g_{\rho\mu}-\partial_\rho g_{\mu\nu}\right). \qquad \text{repeat of (39)} \tag{51}$$

The Riemann tensor, which we will not derive or study in depth herein, is a measure of the curvature of spacetime. (For a purely spatial space like a flat plane or a surface of a sphere, it measures the curvature of space [as opposed to spacetime].) If the spacetime (or space, in purely spatial cases) is flat, the Riemann tensor (50) is zero. If curved, the Riemann tensor is non-zero. There is much math behind all of this, and it is a separate study unto itself.

If the space is flat, the Riemann tensor is zero, and since the Ricci tensor $R_{\mu\eta}$, the Ricci scalar R (49), and the Einstein tensor $G_{\mu\nu}$ are derived from it, those will all also be zero. If curved, all of those quantities will be non-zero.

Each physical case has a different stress-energy tensor $T_{\mu\nu}$ on the right-hand side. The distribution, and motion, of mass-energy in a particular case shows up in a particular expression for $T_{\mu\nu}$. A star has one form for $T_{\mu\nu}$; the entire universe, quite another.

In essence, the metric $g_{\mu\nu}$ (of (51)) is the unknown in (48) that we want to solve for. Given $g_{\mu\nu}$, we basically know everything about the particular physical case we are studying. In principle, knowing the form of $T_{\mu\nu}$, we simply solve for $g_{\mu\nu}$ in (48). In practice, as you may guess from (48) to (51), this is far from trivial.

The usual way problems are handled is to guess at a form for $g_{\mu\nu}$, plug it into (51), plug that result into (50), plug that result into (49), and then, that result into (48). If we then get a balance between the left and right-hand sides of (48), we conclude we have found the correct form of $g_{\mu\nu}$.

Solutions for various cases, such as a spherical astronomical body, the entire universe (given certain assumptions), and accelerating systems have been found that match observation to high degree.

Note, $g_{\mu\nu}$, is generally called the gravitational field. It is a tensor (a matrix in 4D spacetime), unlike the e/m (photon) field A^{μ}, which is a 4-vector.

12.2 Proving Einstein's Equation to be Valid

After deducing (guessing really) (48), Einstein and others needed to show it represents the real world. Einstein himself (I believe) showed his equation applied to a body like the Earth or sun reduced in the Newtonian limit to Newton's law of gravity, That is, in that limit, the Einstein field equation reduces to

$$\mathbf{F} = m\mathbf{a} = -G\frac{mM}{r^2}\mathbf{i}_r \,. \tag{52}$$

We do not go into this analysis herein, but it can be found in almost any general relativity book.

Further, proof, as you have no doubt heard, was the prediction for the bending of starlight as it passed by the sun, which was confirmed by Eddington and his team during a solar eclipse in 1919. That confirmation propelled Einstein into the worldwide limelight.

Shortly thereafter, another confirmation, the perihelion shift of Mercury, was made. And in over a century since, many other confirmations of the theory have been made, via astronomical and cosmological observations, as well as the GPS navigation system.

Problem 11. For the generalized coordinate system of Prob. 3 (polar coordinates), what is the Reimann tensor $R^{i}{}_{jkl}$?

Problem 12. For an inertial frame using Cartesian 3D coordinates and physical clocks at each location to measure time, write down the 4D spacetime metric. What name do we call it? What is the numerical value for each of the components of the Reimann tensor $R^{\alpha}{}_{\beta\gamma\rho}$?

Problem 13. For an inertial frame using cylindrical 3D coordinates and physical clocks at each location to measure time, what is the numerical value for each of the components of the Reimann tensor $R^{\alpha}{}_{\beta\gamma\rho}$?

Problem 14. For an inertial frame using Cartesian 3D coordinates and non-physical clocks at each location that beat twice every two seconds rather than once to measure coordinate (not physical) time, what is the numerical value for each of the components of the Reimann tensor $R^{\alpha}{}_{\beta\gamma\rho}$?

Problem 15. For an inertial frame using cylindrical 3D coordinates and physical or non-physical clocks at each location to measure time, what is the numerical value for each of the components of the Einstein tensor $G_{\mu\nu}$? What then, given the Einstein field equation, must every component in the stress-energy tensor $T_{\mu\nu}$ equal? Does a flat spacetime have to have zero mass-energy?

Problem 17. Follow similar steps as in (5), for the vector **V** expressed as $\mathbf{V} = v_1\,\mathbf{g}^1 + v_2\,\mathbf{g}^2$, to find (53) for that case. Then, can you reason that the relation is also good for non-orthogonal generalized coordinates, any number of dimensions, and spacetime?

$$g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j \,. \tag{53}$$

Problem 18. Use Fig. 2 and the generalized 2D coordinate system shown as part of a 4D spacetime coordinate system, where the third spatial dimension is perpendicular to the 2D plane of that figure and time is just that measured on standard clocks everywhere in the 3D space. Consider a photon emitted at the origin and traveling out along the horizontal direction.
In one second, how many meter sticks does the photon pass? In that same one second, how many grid lines along the x^1 axis does the photon pass? What is the photon's physical speed? What is the photon's coordinate speed (in terms of x^1 per unit time)? Without doing any calculation, what is the spacetime interval Δs for the photon over that second? For any infintesicmal distance the photon may travel, what is the infinitesimal spacetime interval ds? Calculate Δs for the photon over the one second using coordinates x^i and the metric g_{ij}.

13 Appendix: Deducing the Math Form of Christoffel Symbols

13.1 Additional Relations We'll Need

13.1.1 First Relation Needed

We can express g_{ij} as

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \left(\frac{\partial X^m}{\partial x^i}\mathbf{i}_m\right)\cdot\left(\frac{\partial X^r}{\partial x^j}\mathbf{i}_r\right) = \frac{\partial X^m}{\partial x^i}\frac{\partial X^r}{\partial x^j}\delta_{mr} = \frac{\partial X^m}{\partial x^i}\frac{\partial X^m}{\partial x^j}. \tag{54}$$

Problem 19. For the case of Fig. 2, find x^i as a function of X^j, i.e., find $x^i(X^j)$ for i,j = 1,2. Then find X^i as a function of x^j, i.e., find $X^i(x^j)$. Then find $\frac{\partial X^i}{\partial x^j}$ and use it to find g_{ij} for the generalized coordinates of Fig. 2.

Problem 20. For a polar coordinate system, find $x^i(X^j)$ ($x^1 = r$, $x^2 = \phi$). Then find $X^i(x^j)$. Then use the results to determine g_{ij}, via (54), for polar coordinates.

13.1.2 Second Relaion Needed

The following is true, as we will show after stating it.

$$g^{ij} = \frac{\partial x^i}{\partial X^s}\frac{\partial x^j}{\partial X^s}. \tag{55}$$

We know (55) is the inverse of (54). So,

$$g_{ij}g^{jk} = \delta_i^k = \frac{\partial X^m}{\partial x^i}\frac{\partial X^m}{\partial x^j}g^{jk}. \tag{56}$$

The solution to (56) is

$$g^{jk} = \frac{\partial x^j}{\partial X^s}\frac{\partial x^k}{\partial X^s}, \tag{57}$$

as we can see by substituting (57) into (56)

$$g_{ij}g^{jk} = \frac{\partial X^m}{\partial x^i}\frac{\partial X^m}{\partial x^j}\frac{\partial x^j}{\partial X^s}\frac{\partial x^k}{\partial X^s} = \frac{\partial X^m}{\partial x^i}\frac{\partial X^m}{\partial X^s}\frac{\partial x^j}{\partial x^j}\frac{\partial x^k}{\partial X^s} = \frac{\partial X^m}{\partial x^i}\delta_s^m\frac{\partial x^k}{\partial X^s} = \frac{\partial X^m}{\partial x^i}\frac{\partial x^k}{\partial X^m} = \frac{\partial x^k}{\partial x^i} = \delta_i^k. \tag{58}$$

We will use (55) and (57) in what follows.

13.2 Steps We'll Follow

To prove the expression for the Christoffel symbols in terms of the metric (39) (repeated in (51)), we proceed in three steps.

1. Find $\Gamma^k{}_{ij}$ from basis vector relations by which it is defined in (33), then
2. re-express $\Gamma^k{}_{ij}$ from the relation in terms of the metric (39) (repeated in (51)), then
3. compare 1. and 2. to find them equal.

13.2.1 Step 1: Finding the Christoffel Symbols from Their Definition

From the underbrace at the end of the first line of (33), where we generalize by taking $1 \rightarrow j$ and think of i,j,k as 1, …, N for an N dimensional space.

$$\Gamma^k_{i\,j}\mathbf{g}_k = \frac{\partial \mathbf{g}_i}{\partial x^j} \tag{59}$$

We need (27), repeated below in (60),

$$\mathbf{g}_i = \frac{\partial \mathbf{P}}{\partial x^i} \qquad \text{repeat of (27),} \tag{60}$$

which can be written as

$$\mathbf{g}_k = \frac{\partial \mathbf{P}}{\partial x^k} = \frac{\partial}{\partial x^k}\left(X^1\mathbf{i}_1 + X^2\mathbf{i}_2 + \ldots + X^N\mathbf{i}_N\right) = \frac{\partial}{\partial x^k}\left(X^m\mathbf{i}_m\right) = \frac{\partial X^m}{\partial x^k}\mathbf{i}_m\,. \tag{61}$$

Now, we can invert (61) to find the unit basis vector $\mathbf{i}_m$ in terms of the generalized basis vector $\mathbf{g}_k$ by just moving the numerator and denominator on the far right side to the far left side, or more formally, via

$$\frac{\partial x^k}{\partial X^l}\mathbf{g}_k = \frac{\partial x^k}{\partial X^l}\frac{\partial X^m}{\partial x^k}\mathbf{i}_m = \frac{\partial X^m}{\partial X^l}\mathbf{i}_m = \delta^m_l\mathbf{i}_m = \mathbf{i}_l \qquad \rightarrow \qquad \mathbf{i}_l = \frac{\partial x^k}{\partial X^l}\mathbf{g}_k \rightarrow \qquad \mathbf{i}_m = \frac{\partial x^k}{\partial X^m}\mathbf{g}_k\,. \tag{62}$$

Now use (60) in (59) to get

$$\Gamma^k_{i\,j}\mathbf{g}_k = \frac{\partial \mathbf{g}_i}{\partial x^j} = \frac{\partial}{\partial x^j}\frac{\partial \mathbf{P}}{\partial x^i} = \frac{\partial^2}{\partial x^i\partial x^j}\left(X^1\mathbf{i}_1 + X^2\mathbf{i}_2 + \ldots + X^N\mathbf{i}_N\right) = \frac{\partial^2}{\partial x^i\partial x^j}\left(X^l\mathbf{i}_l\right) = \frac{\partial^2 X^l}{\partial x^i\partial x^j}\mathbf{i}_l\,. \tag{63}$$

Then, use (62) in (63) to obtain

$$\Gamma^k_{i\,j}\mathbf{g}_k = \frac{\partial^2 X^l}{\partial x^i\partial x^j}\mathbf{i}_l = \frac{\partial^2 X^l}{\partial x^i\partial x^j}\frac{\partial x^k}{\partial X^l}\mathbf{g}_k\,, \tag{64}$$

from which, we glean

$$\Gamma^k_{i\,j} = \frac{\partial^2 X^l}{\partial x^i\partial x^j}\frac{\partial x^k}{\partial X^l}\,. \tag{65}$$

13.2.2 Step 2: Re-expressing the Christoffel Relations from Those in Terms of the Metric

We stated (39) (repeated in (51)) without proof. If it's true, it must equal (65). So, let's see if it does.

First, we'll convert the Greek sub and super scripts to Roman (spacetime to purely spatial), just to be consistent with the symbols in (65). But, the result will be good for any dimension spaces, with spatial only, or both time and space, coordinates.

$$\Gamma^k_{ij} = \tfrac{1}{2}g^{km}\left(\partial_i g_{jm} + \partial_j g_{mi} - \partial_m g_{ij}\right)\,. \tag{66}$$

We start by using (54) in (66).

$$\Gamma^k_{ij} = \tfrac{1}{2}g^{kp}\left(\frac{\partial}{\partial x^i}\left(\left(\frac{\partial X^m}{\partial x^j}\frac{\partial X^m}{\partial x^p}\right)\right) + \frac{\partial}{\partial x^j}\left(\frac{\partial X^m}{\partial x^p}\frac{\partial X^m}{\partial x^i}\right) - \frac{\partial}{\partial x^p}\left(\frac{\partial X^m}{\partial x^i}\frac{\partial X^m}{\partial x^j}\right)\right) \tag{67}$$

Expanding this, we have

$$\begin{aligned}\Gamma^k_{ij} &= \tfrac{1}{2}g^{kp}\left(\frac{\partial^2 X^m}{\partial x^i\partial x^j}\frac{\partial X^m}{\partial x^p} + \underbrace{\frac{\partial X^m}{\partial x^j}\frac{\partial^2 X^m}{\partial x^i\partial x^p}}_{\text{cancels 5th term}} + \underbrace{\frac{\partial^2 X^m}{\partial x^j\partial x^p}\frac{\partial X^m}{\partial x^i}}_{\text{cancels 6th term}} + \frac{\partial X^m}{\partial x^p}\frac{\partial^2 X^m}{\partial x^j\partial x^i} - \underbrace{\frac{\partial^2 X^m}{\partial x^p\partial x^i}\frac{\partial X^m}{\partial x^j}}_{\text{cancels 2nd term}} - \underbrace{\frac{\partial X^m}{\partial x^i}\frac{\partial^2 X^m}{\partial x^p\partial x^j}}_{\text{cancels 3rd term}}\right) \\ &= g^{kp}\left(\frac{\partial^2 X^m}{\partial x^i\partial x^j}\frac{\partial X^m}{\partial x^p}\right).\end{aligned} \tag{68}$$

Using (57) for g^{kp} gives us

$$\Gamma_{ij}^{k}=\frac{\partial x^{k}}{\partial X^{r}}\frac{\partial x^{p}}{\partial X^{r}}\left(\frac{\partial^{2}X^{m}}{\partial x^{i}\partial x^{j}}\frac{\partial X^{m}}{\partial x^{p}}\right)=\frac{\partial x^{k}}{\partial X^{r}}\frac{\partial^{2}X^{m}}{\partial x^{i}\partial x^{j}}\underbrace{\left(\frac{\partial x^{p}}{\partial X^{r}}\frac{\partial X^{m}}{\partial x^{p}}\right)}_{\delta_{r}^{m}}=\frac{\partial^{2}X^{m}}{\partial x^{i}\partial x^{j}}\frac{\partial x^{k}}{\partial X^{r}}\delta_{r}^{m}=\frac{\partial^{2}X^{m}}{\partial x^{i}\partial x^{j}}\frac{\partial x^{k}}{\partial X^{m}}. \tag{69}$$

Changing the dummy index *m* into *l* in (69) gives us

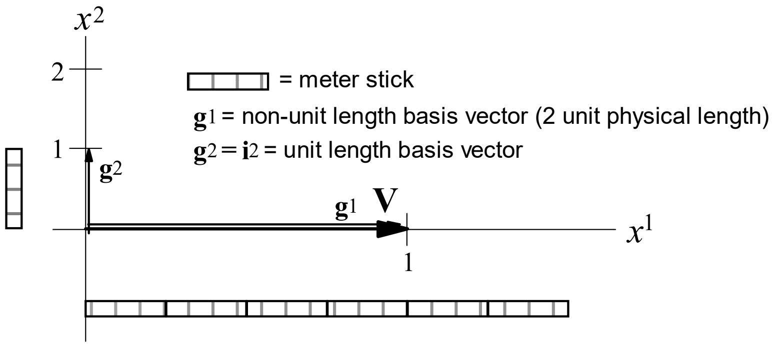
$$\Gamma_{ij}^{k}=\frac{\partial^{2}X^{l}}{\partial x^{i}\partial x^{j}}\frac{\partial x^{k}}{\partial X^{l}}. \tag{70}$$

13.2.3 Step 3: Comparing Results

(70) equals (65), telling us that (66) is indeed the expression for the Christoffel symbols in terms of the metric. QED.

Solutions to Problems

Problem 1: In Fig. 2, we stretched our horizontal axis by a factor of 2. Draw the equivalent of Fig. 2, if the stretching it were a factor of 4, instead. What are the coordinates in the new coordinate system of the same vector **V**? How long, in terms of meter sticks would the new basis vector $\mathbf{g}_1$ be in order to keep $v^1\mathbf{g}_1$ a vector with length equal to the same number of meter sticks? Repeat all of the steps of equation (2) for this new coordinate system.



$$\mathbf{V} = v^1\mathbf{g}_1 + v^2\,\mathbf{g}_2 = 1\mathbf{g}_1 + 0\,\mathbf{g}_2 \qquad v^i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\mathbf{g}_1$ has length of 4 meter sticks

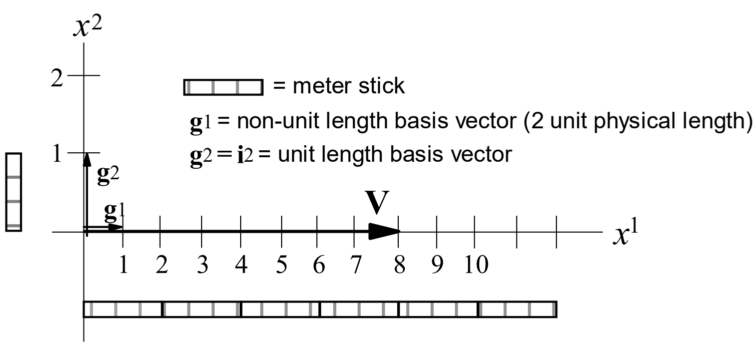
$|\mathbf{V}| = |1\ \mathbf{g}_1| = |\mathbf{g}_1| = 4$ meters

$$|\mathbf{V}|^2 = \mathbf{V}\cdot\mathbf{V} = (v^1\mathbf{g}_1)\cdot(v^1\mathbf{g}_1) + (v^2\mathbf{g}_2)\cdot(v^2\mathbf{g}_2) = v^1v^1\underbrace{(\mathbf{g}_1\cdot\mathbf{g}_1)}_{\text{label } g_{11}} + v^2v^2\underbrace{(\mathbf{g}_2\cdot\mathbf{g}_2)}_{\text{label } g_{22}}\ \left(\neq v^1v^1 + v^2v^2\right)$$

$$= g_{11}v^1v^1 + g_{22}v^2v^2 = g_{ij}v^iv^j = (16)v^1v^1 + (1)v^2v^2 = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 16 & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 16 \\ 0 \end{bmatrix} = 16$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} g_{11} & \\ & g_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} v^1 & v^2 \end{bmatrix}\begin{bmatrix} g_{11} & \\ & g_{22} \end{bmatrix}\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = g_{ij}v^iv^j \qquad \text{where } g_{ij} = \mathbf{g}_i\cdot\mathbf{g}_j.$$

Problem 2: Draw the equivalent of Fig. 2, if the stretching it were a factor of ½. What are the coordinates in the new coordinate system of the same vector **V**? How long, in terms of meter sticks would the new basis vector $\mathbf{g}_1$ be in order to keep $v^1\mathbf{g}_1$ a vector with length equal to the same number of meter sticks? Repeat all of the steps of equation (2) for this new coordinate system.



$$\mathbf{V} = v^1\mathbf{g}_1 + v^2\,\mathbf{g}_2 = 8\mathbf{g}_1 + 0\,\mathbf{g}_2 \qquad v^i = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

$\mathbf{g}_1$ has length of ½ meter stick

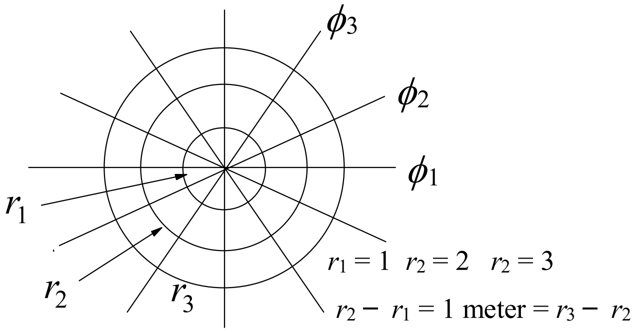
$|\mathbf{V}| = |8\ \mathbf{g}_1| = 8|\mathbf{g}_1| = 4$ meters

$$|\mathbf{V}|^2 = \mathbf{V}\cdot\mathbf{V} = (v^1\mathbf{g}_1)\cdot(v^1\mathbf{g}_1) + (v^2\mathbf{g}_2)\cdot(v^2\mathbf{g}_2) = v^1v^1\underbrace{(\mathbf{g}_1\cdot\mathbf{g}_1)}_{\text{label } g_{11}} + v^2v^2\underbrace{(\mathbf{g}_2\cdot\mathbf{g}_2)}_{\text{label } g_{22}}\ \left(\neq v^1v^1 + v^2v^2\right)$$

$$= g_{11}v^1v^1 + g_{22}v^2v^2 = g_{ij}v^iv^j = \left(\tfrac{1}{4}\right)v^1v^1 + (1)v^2v^2 = \begin{bmatrix} 8 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{4} & \\ & 1 \end{bmatrix}\begin{bmatrix} 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = 16 \qquad (71)$$

$$= \begin{bmatrix} 8 & 0 \end{bmatrix}\begin{bmatrix} g_{11} & \\ & g_{22} \end{bmatrix}\begin{bmatrix} 8 \\ 0 \end{bmatrix} = \begin{bmatrix} v^1 & v^2 \end{bmatrix}\begin{bmatrix} g_{11} & \\ & g_{22} \end{bmatrix}\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = g_{ij}v^iv^j \qquad \text{where } g_{ij} = \mathbf{g}_i\cdot\mathbf{g}_j.$$

Problem 3. A polar coordinate system is a generalized (non-Cartesian) coordinate system with $x^i = [x^1, x^2]^T = [r, \phi]^T$. Sketch the gird lines for this system. Express the line element for it. From that, deduce the metric. Does it depend on location in space?



$$ds^2 = dr^2 + r^2 d\phi^2 \qquad g_{ij} = \begin{bmatrix} 1 & \\ & r^2 \end{bmatrix}$$

Yes, it depends on the coordinate r.

Problem 4. Express (43) in terms of the 4-velocity u^μ.

$$u^\mu = \frac{dx^\mu}{d\tau} \qquad \rightarrow \qquad \frac{du^\mu}{d\tau} + \Gamma^\mu_{\rho\sigma} u^\rho u^\sigma = 0$$

Problem 5. Determine the Christoffel symbols in a Minkowski coordinate system (special case of generalized coordinates, with Cartesian coordinates in 3D and physical time (times c) as the 4$^{\text{th}}$ coordinate). Explain how you got your answer.

From (51), repeated below

$$\text{Christoffel symbol} = \Gamma^\sigma_{\mu\nu} = \tfrac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) \qquad (51)$$

and the Minkowski metric (which has all constant components)

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad \text{metric symbol for Minkowski coordinates, } g_{\mu\nu} = \eta_{\mu\nu} \,,$$

we see that all derivatives of all components in the metric are zero. So, in (51),

$$\Gamma^\sigma_{\mu\nu} = 0 \quad \text{in Minkowski coordinates in inertial frame .}$$

Problem 6. Express (47) in terms of the 4-velocity u^μ, and multiply both sides by m. Does this look like what you (hopefully) learned for the equation of motion for a force on a charged particle due to an e/m field? (The quantity on the RHS is the Lorentz force on a moving particle of charge q [magnetic force on a current] plus the electric force from a static electric field.) Express the result at for low velocities ($v << c$), in flat spacetime (Minkowski metric), for only the 3D part of the velocity. Look familiar?

$$m\left(\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\right) = qF^{\mu\beta}u_\beta \qquad F^{\mu\beta} = \text{ electromagnetic field tensor} \qquad (72)$$

For non-relativistic speeds, where $\tau \rightarrow t$, and for a Minkowski coordinate system (where Christoffel symbols are zero), for motion in 3D,

$$m\frac{d^2x^i}{dt^2} = ma^i = qF^{i\beta}v_\beta = f^i = \text{e/m force on a particle .} \qquad (73)$$

Problem 7. Using what you learned from Prob. 3, find g^{ij} for a polar coordinate system. Then, for a vector $v^i = [3, 2]^T$ find the covariant components v_i in terms of r. For that vector, show that its length squared is given by both $g_{ij}v^iv^j$ and $g^{ij}v_iv_j$.

$$g^{ij} \text{ is the inverse of } g_{ij}. \quad \text{So, } g^{ik}g_{kj} = \delta^i_j \rightarrow \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} = \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix}\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix}\begin{bmatrix} 1 & \\ & r^2 \end{bmatrix} = \begin{bmatrix} g^{11} & g^{12}r^2 \\ g^{21} & g^{22}r^2 \end{bmatrix} \tag{74}$$

Equating the four components in the identity matrix with those of the last matrix, we find

$$g^{11} = 1 \qquad g^{12} = g^{21} = 0 \qquad g^{22} = \tfrac{1}{r^2}$$

$$g^{ij} = \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} = \begin{bmatrix} 1 & \\ & \frac{1}{r^2} \end{bmatrix} \tag{75}$$

It should be immediately obvious that (75) times the matrix g_{ij} gives us the identity matrix. The covariant components are

$$v_i = g_{ij}v^j \qquad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & \\ & r^2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2r^2 \end{bmatrix} \tag{76}$$

To find magnitude squared of the vector two ways,

$$|\mathbf{v}|^2 = g_{ij}v^iv^j = v_iv^i = \begin{bmatrix} 3 & 2r^2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 9 + 4r^2$$

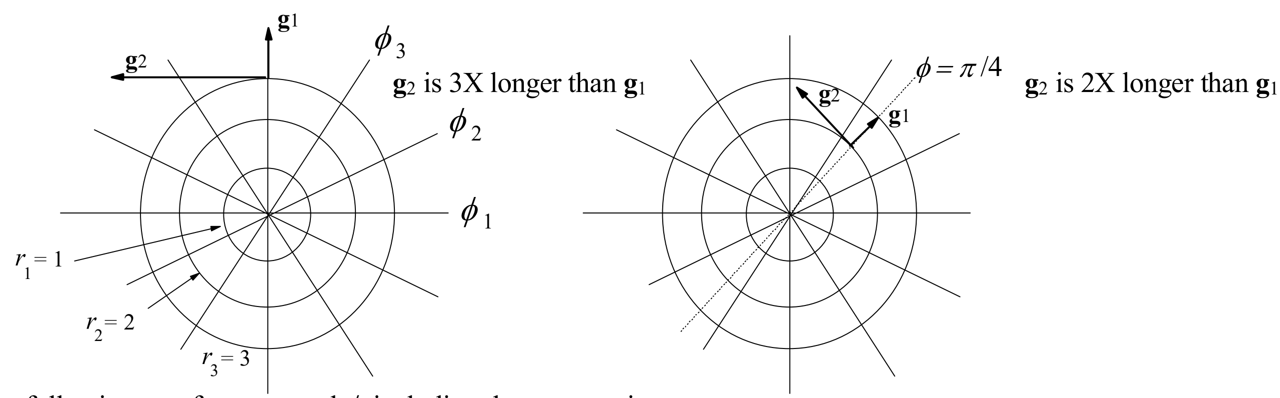
$$|\mathbf{v}|^2 = g^{ij}v_iv_j = \begin{bmatrix} 3 & 2r^2 \end{bmatrix}\begin{bmatrix} 1 & \\ & \frac{1}{r^2} \end{bmatrix}\begin{bmatrix} 3 \\ 2r^2 \end{bmatrix} = \begin{bmatrix} 3 & 2r^2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 9 + 4r^2 \tag{77}$$

Problem 8. Using what you learned from Sect. 9, find $\mathbf{g}_1 = \mathbf{g}_r$ and $\mathbf{g}_2 = \mathbf{g}_\phi$ for polar coordinates. Note that in the Cartesian system $\mathbf{P} = X^1\mathbf{i}_1 + X^2\mathbf{i}_2 = r\cos\phi\,\mathbf{i}_1 + r\sin\phi\,\mathbf{i}_2$. In the figure you drew for Prob. 3, at the point $r = 3$ and $\phi = \pi/2$, draw in $\mathbf{g}_1 = \mathbf{g}_r$ and $\mathbf{g}_2 = \mathbf{g}_\phi$. At the point $r = 2$ and $\phi = \pi/4$, draw in $\mathbf{g}_1 = \mathbf{g}_r$ and $\mathbf{g}_2 = \mathbf{g}_\phi$. Does $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ in both cases? If so, show it.

$$\mathbf{g}_i = \frac{\partial\mathbf{P}}{\partial x^i} = \frac{\partial}{\partial x^i}\left(X^1\mathbf{i}_1 + X^2\mathbf{i}_2\right) = \frac{\partial}{\partial x^i}\left(r\cos\phi\,\mathbf{i}_1 + r\sin\phi\,\mathbf{i}_2\right)$$

$$\mathbf{g}_1 = \mathbf{g}_r = \frac{\partial\mathbf{P}}{\partial r} = \frac{\partial}{\partial r}\left(r\cos\phi\,\mathbf{i}_1 + r\sin\phi\,\mathbf{i}_2\right) = \cos\phi\,\mathbf{i}_1 + \sin\phi\,\mathbf{i}_2 \tag{78}$$

$$\mathbf{g}_2 = \mathbf{g}_\phi = \frac{\partial\mathbf{P}}{\partial\phi} = \frac{\partial}{\partial\phi}\left(r\cos\phi\,\mathbf{i}_1 + r\sin\phi\,\mathbf{i}_2\right) = -r\sin\phi\,\mathbf{i}_1 + r\cos\phi\,\mathbf{i}_2$$



Yes. The following true for any r and ϕ, including these two points.

$$\mathbf{g}_1 \cdot \mathbf{g}_1 = \left(\cos\phi\,\mathbf{i}_1 + \sin\phi\,\mathbf{i}_2\right)\cdot\left(\cos\phi\,\mathbf{i}_1 + \sin\phi\,\mathbf{i}_2\right) = \cos^2\phi + \sin^2\phi = 1 \text{ everywhere} = g_{11}$$

$$\mathbf{g}_1 \cdot \mathbf{g}_2 = \left(\cos\phi\,\mathbf{i}_1 + \sin\phi\,\mathbf{i}_2\right)\cdot\left(-r\sin\phi\,\mathbf{i}_1 + r\cos\phi\,\mathbf{i}_2\right) = -r\cos\phi\sin\phi + r\sin\phi\cos\phi = 0 = g_{12} = g_{21} \tag{79}$$

$$\mathbf{g}_2 \cdot \mathbf{g}_2 = \left(-r\sin\phi\,\mathbf{i}_1 + r\cos\phi\,\mathbf{i}_2\right)\cdot\left(-r\sin\phi\,\mathbf{i}_1 + r\cos\phi\,\mathbf{i}_2\right) = r^2\sin^2\phi + r^2\cos^2\phi = r^2$$

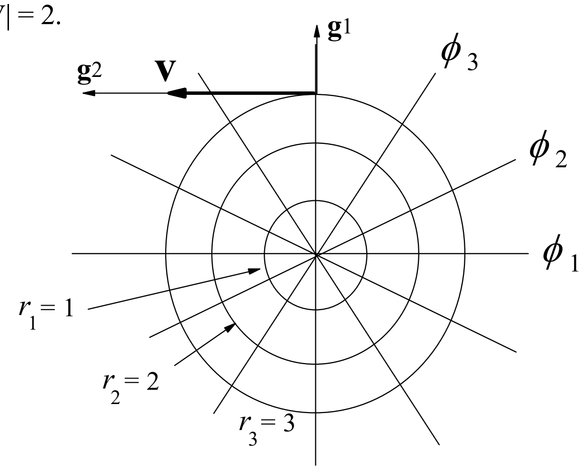
$$g_{ij} = \begin{bmatrix} 1 & \\ & r^2 \end{bmatrix}. \tag{80}$$

Problem 9. Consider a polar coordinate system centered on a vortex (like a hurricane or water swirling down a sink drain) and assume we live in a Newtonian (not relativistic) world. The physically measured velocity of the fluid is dependent on position. Fluid molecules far from the center travel at slower speeds than molecules close to the center. The generalized coordinates of velocity depend on location, i.e., on r and ϕ, expressed as $v^i = v^i(r, \phi) = v^i(x^1, x^2)$. At the point $r = 3$ and $\phi = \pi/2$ velocity is, in polar coordinates, $v^i(3, \pi/2) = [0, 2/3]^T = 0\,\mathbf{g}_r + 2/3\,\mathbf{g}_\phi$. What direction in space does this vector point in? Using g_{ij} find the magnitude of this vector (its speed). Draw this vector in the figure you drew for Prob. 3.

It points in the direction of $\mathbf{g}_2$ in the LH figure of the previous problem solution, i.e., to the left horizontally.

$$|\mathbf{V}|^2 = g_{ij}v^iv^j = \begin{bmatrix}0 & 2/3\end{bmatrix}\begin{bmatrix}1 & \\ & r^2\end{bmatrix}\begin{bmatrix}0\\2/3\end{bmatrix} \xrightarrow[\text{point}]{\text{at this}} = \begin{bmatrix}0 & 2/3\end{bmatrix}\begin{bmatrix}1 & \\ & 3^2\end{bmatrix}\begin{bmatrix}0\\2/3\end{bmatrix} = \begin{bmatrix}0 & 2/3\end{bmatrix}\begin{bmatrix}0\\2\times 3\end{bmatrix} = 4 \quad (81)$$

So, its magnitude (speed) $|\mathbf{V}| = 2$.

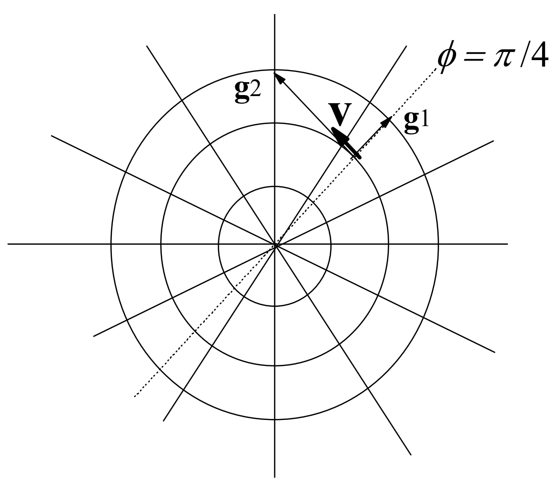


Problem 10. For the same vortex of Prob. 9, and the second point ($r = 2$ and $\phi = \pi/4$) of Prob. 8, the velocity, in generalized coordinates is $v^i(2, \pi/4) = [0, 1/3]^T = 0\,\mathbf{g}_r + 1/3\,\mathbf{g}_\phi$ What direction in space does this vector point in? Using g_{ij} find the magnitude of this vector (its speed). Draw this vector in the figure you drew for Prob. 3.

Since the velocity vector points in the direction of $\mathbf{g}_2 = \mathbf{g}_\phi$ it is perpendicular to the radial line at the point. The point is at 45° from the right side of the horizontal axis (from $\phi = 0$). So, the vector points at 45° + 90° = 135° from the $\phi = 0$ line, i.e., up and to the left.

$$|\mathbf{v}|^2 = g_{ij}v^iv^j = \begin{bmatrix}0 & 1/3\end{bmatrix}\begin{bmatrix}1 & \\ & r^2\end{bmatrix}\begin{bmatrix}0\\1/3\end{bmatrix} \xrightarrow[\text{point}]{\text{at this}} = \begin{bmatrix}0 & 1/3\end{bmatrix}\begin{bmatrix}1 & \\ & 2^2\end{bmatrix}\begin{bmatrix}0\\1/3\end{bmatrix} = \begin{bmatrix}0 & 1/3\end{bmatrix}\begin{bmatrix}0\\4/3\end{bmatrix} = \frac{4}{9} \quad (82)$$

So, the speed $|\mathbf{V}| = 2/3$.



Problem 11. For the generalized coordinate system of Prob. 3 (polar coordinates), what is the Reimann tensor R^i_{jkl}?

Without doing any calculations, we know the Riemann tensor must be zero, because the space is flat. $R^i_{jkl} = 0$ for all i,j,k,l (each equal either 1 or 2 since this is a 2D problem).

Problem 12. For an inertial frame using Cartesian 3D coordinates and physical clocks at each location to measure time, write down the 4D spacetime metric. What name do we call it? What is the numerical value for each of the components of the Reimann tensor $R^{\alpha}{}_{\beta\gamma\rho}$?

$$g_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \eta_{\mu\nu} \quad \text{Called the Minkowski metric (or the Lorentz metric, sometimes).} \tag{83}$$

All Christoffel symbols (51) are zero, since $\eta_{\mu\nu}$ is constant, and derivatives of a constant are zero. So, from (50), the Riemann tensor is zero (which we knew anyway, since the Minkowski metric is only good in flat spacetime [inertial reference frames]).

Problem 13. For an inertial frame using cylindrical 3D coordinates and physical clocks at each location to measure time, what is the numerical value for each of the components of the Reimann tensor $R^{\alpha}{}_{\beta\gamma\rho}$?

Even though the metric in a cylindrical system is not constant, we know the Riemann tensor must still (if we did all the extensive math) equal zero, because the reference frame is inertial.

Note: A metric with all constant components must have the Riemann tensor = zero (because all of the Christoffel symbols = 0). But, a metric that does not have all constant components can still have Riemann tensor = zero. A flat space can have a Cartesian/Minkowski metric (all constant components) or not (variable components). But, in either case, its curvature is zero, because it's flat.

A curved space can never have a metric with all constant components (because then, the Riemann tensor has to be zero, and it can't be for a curved space). So, a curved space can never have a Cartesian/Minkowski metric.

Problem 14. For an inertial frame using Cartesian 3D coordinates and non-physical clocks at each location that beat twice every two seconds rather than once to measure coordinate (not physical) time, what is the numerical value for each of the components of the Riemann tensor $R^{\alpha}{}_{\beta\gamma\rho}$?

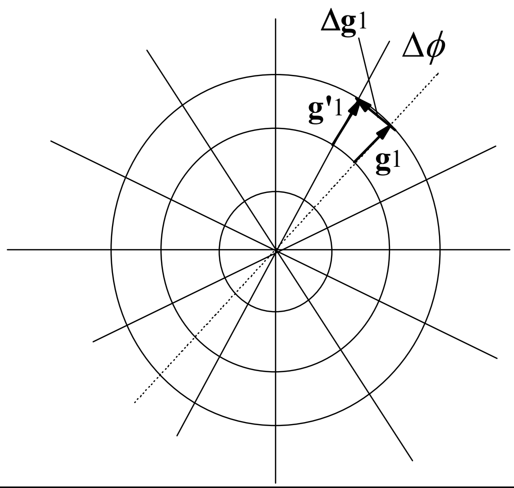
The Riemann tensor is still zero, because in the spacetime is flat. No need to do any calculations.

Problem 15. For an inertial frame using cylindrical 3D coordinates and physical or non-physical clocks at each location to measure time, what is the numerical value for each of the components of the Einstein tensor $G_{\mu\nu}$? What then, given the Einstein field equation, must every component in the stress-energy tensor $T_{\mu\nu}$ equal? Does a flat spacetime have to have zero mass-energy?

Since the Riemann tensor must equal zero in flat spacetime (an inertial system), the Ricci tensor (49), which is a contraction of the Riemann tensor, must be zero. And the Ricci scalar R (49), which is a contraction of the Ricci tensor, must be zero. So, the Einstein tensor $G_{\mu\nu}$ (48) which is made up of the Ricci tensor and the Ricci scalar, must equal zero.

Then, in Einstein's field equation (48), the left side is zero, so the right side must also be zero. So, stress-energy tensor $T_{\mu\nu}$ = 0, which means there is no mass-energy. That, we know, means the spacetime must be flat, since mass-energy is what curves spacetime. But, that is what we started with, an inertial frame (flat 4D), so it makes sense.

Problem 16. Make a sketch showing, in polar coordinates, how the $\mathbf{g}_r$ basis vector (think $\mathbf{g}_1$ in (33) changes direction as we vary ϕ. In other words, the derivative of that vector with respect to ϕ is not simply a change along the original direction of that vector. So, we can't expect both terms in (34) to be simply numbers multiplied by the original vector $\mathbf{g}_r$. The derivative must incorporate other directions, i.e., other $\mathbf{g}_i$. And (34) does just that. The Christoffel symbols tell us how much of each other basis vector plays as part of the change in $\mathbf{g}_r$.



Problem 17. Follow similar steps as in (5), for the vector **V** expressed as $\mathbf{V} = v_1\,\mathbf{g}^1 + v_2\,\mathbf{g}^2$, to find (53) for that case. Then, can you reason that the relation is also good for non-orthogonal generalized coordinates, any number of dimensions, and spacetime?

$$g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j \quad \text{repeat of (53)}$$

$$\mathbf{V} = v_1\mathbf{g}^1 + v_2\mathbf{g}^2 \text{ at the point where } \mathbf{V} \text{ is.}$$

$$|\mathbf{V}|^2 = \mathbf{V}\cdot\mathbf{V} = \left(v_1\mathbf{g}^1\right)\cdot\left(v_1\mathbf{g}^1\right) + \left(v_2\mathbf{g}^2\right)\cdot\left(v_2\mathbf{g}^2\right) = v_1v_1\underbrace{\left(\mathbf{g}^1\cdot\mathbf{g}^1\right)}_{\text{label } g^{11}} + v_2v_2\underbrace{\left(\mathbf{g}^2\cdot\mathbf{g}^2\right)}_{\text{label } g^{22}} \tag{84}$$

$$= g^{11}v_1v_1 + g^{22}v_2v_2 = g^{ij}v_iv_j \qquad \text{where } \boxed{g^{ij} = \mathbf{g}^i\cdot\mathbf{g}^j}.$$

Problem 18. Use Fig. 2 and the generalized 2D coordinate system shown as part of a 4D spacetime coordinate system, where the third spatial dimension is perpendicular to the 2D plane of that figure and time is just that measured on standard clocks everywhere in the 3D space. Consider a photon emitted at the origin and traveling out along the horizontal direction.
In one second, how many meter sticks does the photon pass? In that same one second, how many grid lines along the x^1 axis does the photon pass? What is the photon's physical speed? What is the photon's coordinate speed (in terms of x^1 per unit time)? Without doing any calculation, what is the spacetime interval Δs for the photon over that second? For any infinitesimal distance the photon may travel, what is the infinitesimal spacetime interval ds? Calculate Δs for the photon over the one second using coordinates x^μ and the metric $g_{\mu\nu}$.

3 X 10^8 meters. 1.5 X 10^8 grid lines. 3 X 10^8 meters/second. 1.5 X 10^8 grid lines/second. $\Delta s = 0$, as it must be for any photon anywhere. $ds = 0$. Note that $\Delta x^0 = c\Delta t$, and here $\Delta t = 1$.

$$(\Delta s)^2 = g_{\mu\nu}\Delta x^\mu \Delta x^\nu = \begin{bmatrix} c & 1.5\times10^8 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & & & \\ & 4 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} c \\ 1.5\times10^8 \\ 0 \\ 0 \end{bmatrix} \tag{85}$$

$$= -c^2 + 4\left(1.5\times10^8\right)^2 = -\left(3\times10^8\right)^2 + \left(3\times10^8\right)^2 = 0.$$

Problem 19. For the case of Fig. 2, find x^i as a function of X^j, i.e., find $x^i(X^j)$ for $i,j = 1,2$. Then find X^i as a function of x^j, i.e., find $X^i(x^j)$. Then find $\frac{\partial X^i}{\partial x^j}$ and use it to find g_{ij} for the generalized coordinates of Fig. 2.

$$x^1 = \tfrac{1}{2}X^1 = x^1\left(X^1, X^2\right) \qquad x^2 = X^2 = x^2\left(X^1, X^2\right)$$
$$X^1 = 2x^1 = X^1\left(x^1, x^2\right) \qquad X^2 = x^2 = X^2\left(x^1, x^2\right) \tag{86}$$

$$\frac{\partial X^1}{\partial x^1} = 2 \qquad \frac{\partial X^1}{\partial x^2} = 0 \qquad \frac{\partial X^2}{\partial x^1} = 0 \qquad \frac{\partial X^2}{\partial x^2} = 1 \qquad\qquad g_{ij} = \frac{\partial X^m}{\partial x^i}\frac{\partial X^m}{\partial x^j}$$

$$g_{11} = \frac{\partial X^1}{\partial x^1}\frac{\partial X^1}{\partial x^1} + \frac{\partial X^2}{\partial x^1}\frac{\partial X^2}{\partial x^1} = 2^2 + 0 = 4 \qquad g_{12} = \frac{\partial X^1}{\partial x^1}\frac{\partial X^1}{\partial x^2} + \frac{\partial X^2}{\partial x^1}\frac{\partial X^2}{\partial x^2} = 2\cdot 0 + 0\cdot 1 = 0 \tag{87}$$

$$g_{21} = \frac{\partial X^1}{\partial x^2}\frac{\partial X^1}{\partial x^1} + \frac{\partial X^2}{\partial x^2}\frac{\partial X^2}{\partial x^1} = 0 \qquad g_{22} = \frac{\partial X^1}{\partial x^2}\frac{\partial X^1}{\partial x^2} + \frac{\partial X^2}{\partial x^2}\frac{\partial X^2}{\partial x^2} = 0 + 1^2 = 1$$

$$g_{ij} = \begin{bmatrix} 4 & \\ & 1 \end{bmatrix} \tag{88}$$

Problem 20. For a polar coordinate system, find $x^i(X^j)$ ($x^1 = r$, $x^2 = \phi$). Then find $X^i(x^j)$. Then use the results to determine g_{ij}, via (54), for polar coordinates.

$$x^1 = r = \sqrt{\left(X^1\right)^2 + \left(X^2\right)^2} \qquad x^2 = \theta = tan^{-1}\left(\frac{X^2}{X^1}\right)$$
$$X^1 = r\,cos\,\theta \qquad X^2 = r\,sin\,\theta \tag{89}$$

$$\frac{\partial X^1}{\partial x^1} = \frac{\partial r\,cos\,\theta}{\partial r} = cos\,\theta \qquad \frac{\partial X^1}{\partial x^2} = \frac{\partial r\,cos\,\theta}{\partial \theta} = -r\,sin\,\theta \qquad \frac{\partial X^2}{\partial x^1} = \frac{\partial r\,sin\,\theta}{\partial r} = sin\,\theta \qquad \frac{\partial X^2}{\partial x^2} = \frac{\partial r\,sin\,\theta}{\partial \theta} = r\,cos\,\theta$$

$$g_{ij} = \frac{\partial X^m}{\partial x^i}\frac{\partial X^m}{\partial x^j} \tag{90}$$

$$g_{11} = g_{rr} = cos^2\,\theta + sin^2\,\theta = 1 \qquad g_{12} = cos\,\theta\left(-r\,sin\,\theta\right) + sin\,\theta\; r\,cos\,\theta = 0$$
$$g_{21} = -r\,sin\,\theta\,cos\,\theta + r\,cos\,\theta\,sin\,\theta = 0 \qquad g_{22} = \left(-r\,sin\,\theta\right)^2 + \left(r\,cos\,\theta\right)^2 = r^2\left(cos^2\,\theta + sin^2\,\theta\right) = r^2$$

$$g_{ij} = \begin{bmatrix} 1 & \\ & r^2 \end{bmatrix} \tag{91}$$